2 Probability

2.1 – 2.6

Chapter Outline

2

2.1 Sample Space

2.2 Events

2.3 Counting Sample Points

2.4 Probability of an Event

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2.6 Conditional Probability, Independence, and the Product Rule

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2.1 Sample Space

- Observation: Any recording of information , whether it be numerical or categorical

Example: {N, D, N, N, D}, N = non-defective, D = defective, n=5

- Experiment: Any process that generates a set of data.

Example: tossing a coin: {head, tail}

launching a missile and observing its velocity with time

- A Sample Space: The set of all possible outcomes of a statistical experiment. It is represented by the symbol *S*.
- A sample point: each outcome in a sample space (element or member).

Sample Space

- S is discrete if it consists of a finite or countable infinite set of outcomes.
- S is continuous if it contains an interval of real numbers.
- Example: Randomly select a camera and record the recycle time of a flash. $S = R^+ = \{x \mid x > 0\}$, the positive real numbers.
  - Suppose it is known that all recycle times are between 1.5 and 5 seconds. Then

    $S = \{x \mid 1.5 < x < 5\}$ is continuous.
  - It is known that the recycle time has only three values(low, medium or high). Then $S = \{low, medium, high\}$ is discrete.
  - Does the camera conform to minimum recycle time specifications?

    $S = \{yes, no\}$ is discrete.

Sample Space

- Example: Tossing a dice: the sample space for the number on the top face: S_1 = {1, 2, 3, 4, 5, 6}
- Example: Tossing a dice: the sample space for whether the number is even or odd: S_2 = {even, odd}
- An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a dice is tossed once.
- S = {HH, HT, T1, T2, T3, T4, T5, T6}.
- Sample spaces with large or infinite number of elements are best described by a statement or rule.

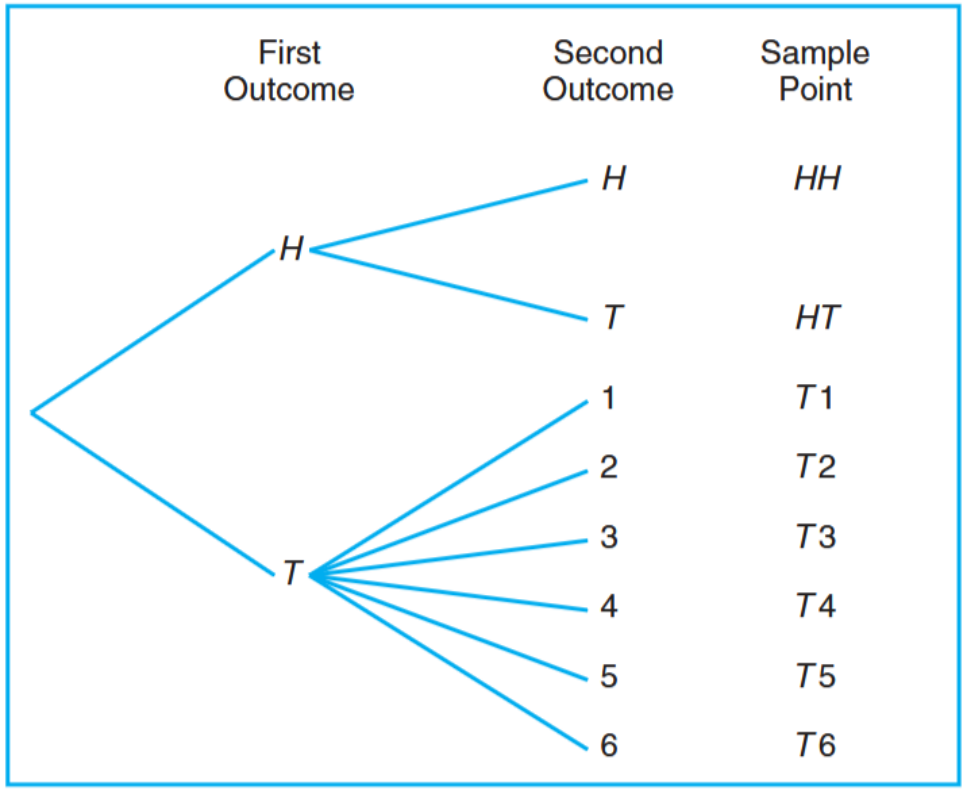
  Example: The set of all high school students who got 75% or more:

  S = {x|x is a student with an average over 75%}

  (S is the set of all x such that x is a student with an average over 75%)

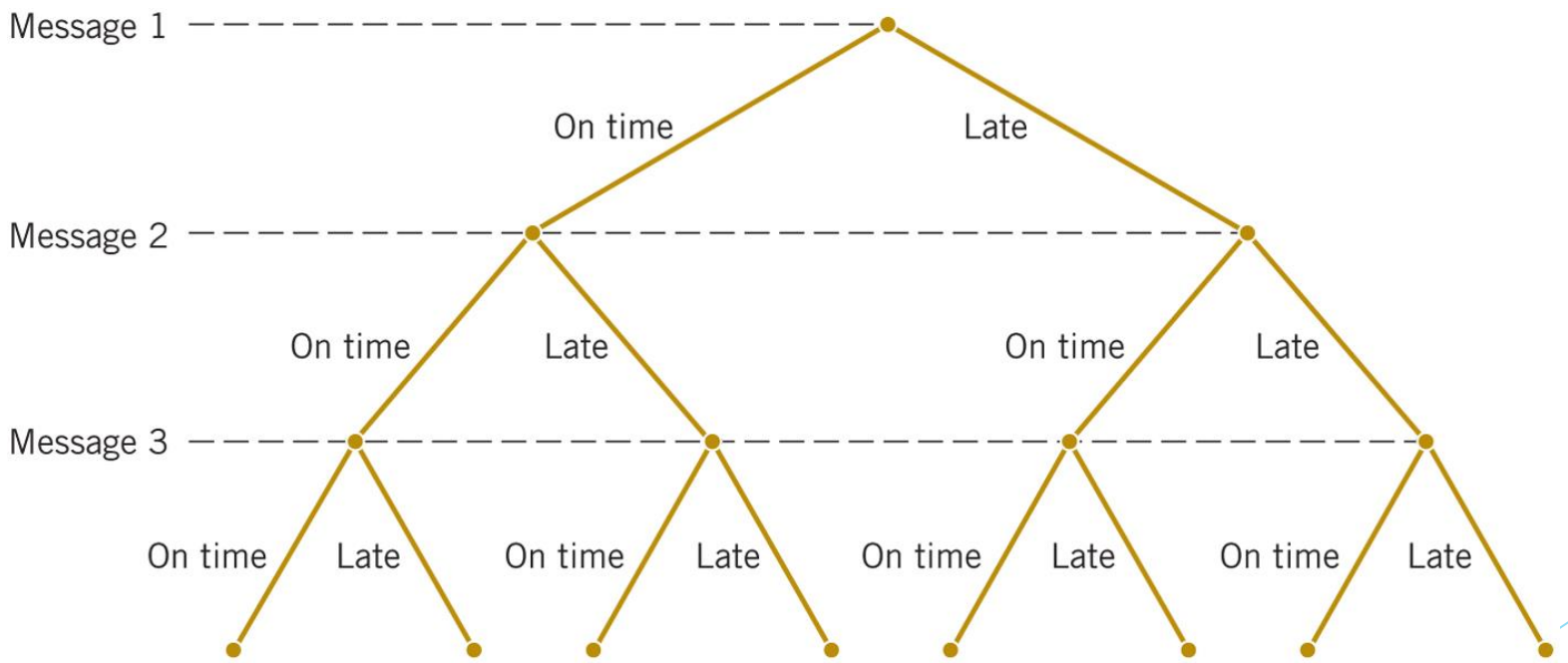
Tree Diagram

- Tree Diagram: Systematic list of sample space elements.
- An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a dice is tossed once.



Sample Space Defined By A Tree Diagram

- Example 2-3: Messages are classified as on-time(o) or late(l). Classify the next 3 messages.
- *S* = {ooo, ool, olo, oll, loo, lol, llo, lll}



Example 2-4 **Automobile Options** An automobile manufacturer provides vehicles equipped with selected options. Each vehicle is ordered

- With or without an automatic transmission
- With or without a sunroof
- With one of three choices of a stereo system
- With one of four exterior colors

If the sample space consists of the set of all possible vehicle types, what is the number of outcomes in the sample space? The sample space contains 48 outcomes. The tree diagram for the different types of vehicles is displayed in Fig. 2-6.

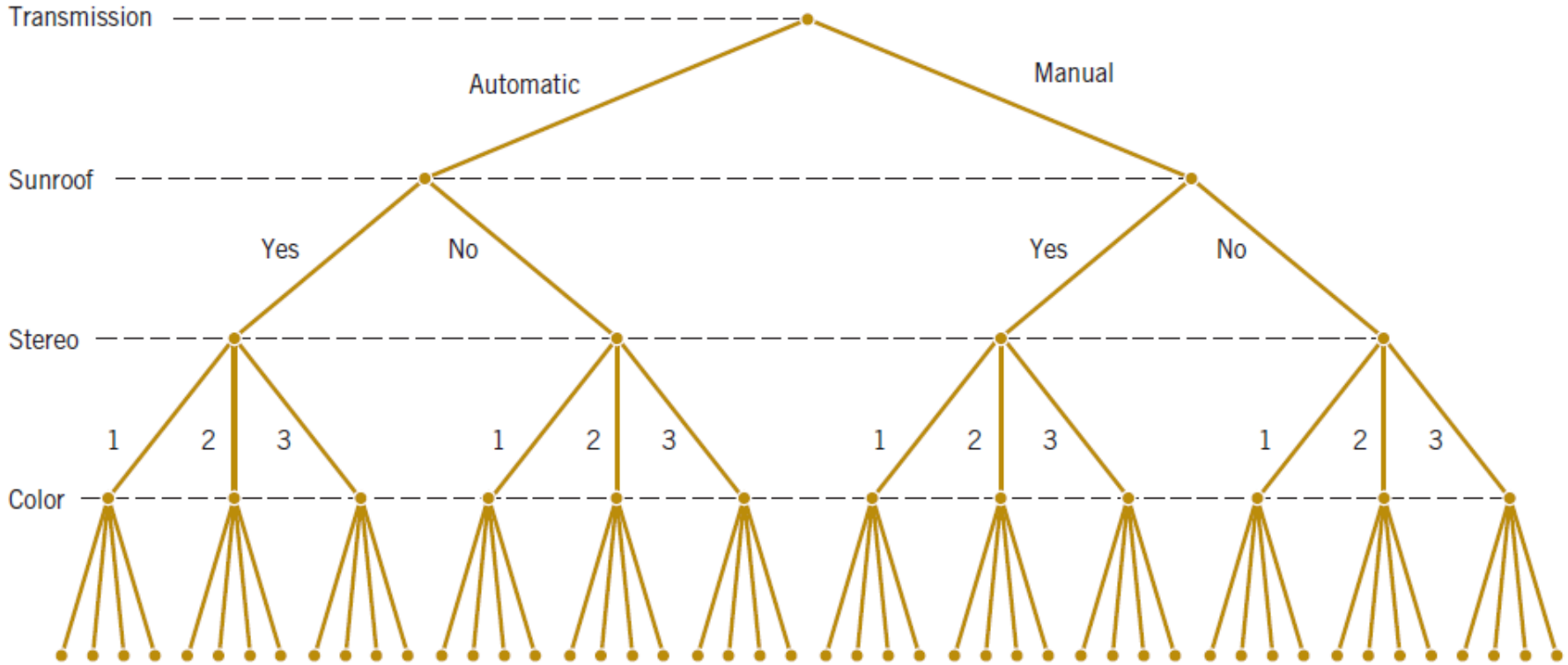


FIGURE 2-6 Tree diagram for different types of vehicles with 48 outcomes in the sample space.

2.2 Events

- An event (E) is a subset of the sample space of a random experiment.
- Event A with outcome of dice tossing which is divisible by 2:

  $\boldsymbol{A = \{2, 4, 6\}}$

- Event combinations
  - The Union of two events consists of all outcomes that are contained in one event or the other, denoted as $E_1 \cup E_2$.
  - The Intersection of two events consists of all outcomes that are contained in one event and the other, denoted as $E_1 \cap E_2$.
  - The Complement of an event is the set of outcomes in the sample space that are not contained in the event, denoted as **(E', $\overline{E}$ or E^c)**

Discrete Events

Suppose that the recycle times of two cameras are recorded. Consider only whether the cameras conform to the manufacturing specifications. We abbreviate *yes* and *no* as *y* and *n*. The sample space is S = {*yy, yn, ny, nn*}.

Suppose, E_1 denotes an event that at least one camera conforms to specifications, then

Suppose, E_2 denotes an event that no camera conforms to specifications,

Suppose, E_3 denotes an event that at least one camera does not conform.

Continuous Events

Measurements of the thickness of a part are modeled with the sample space: $S = R^+$.

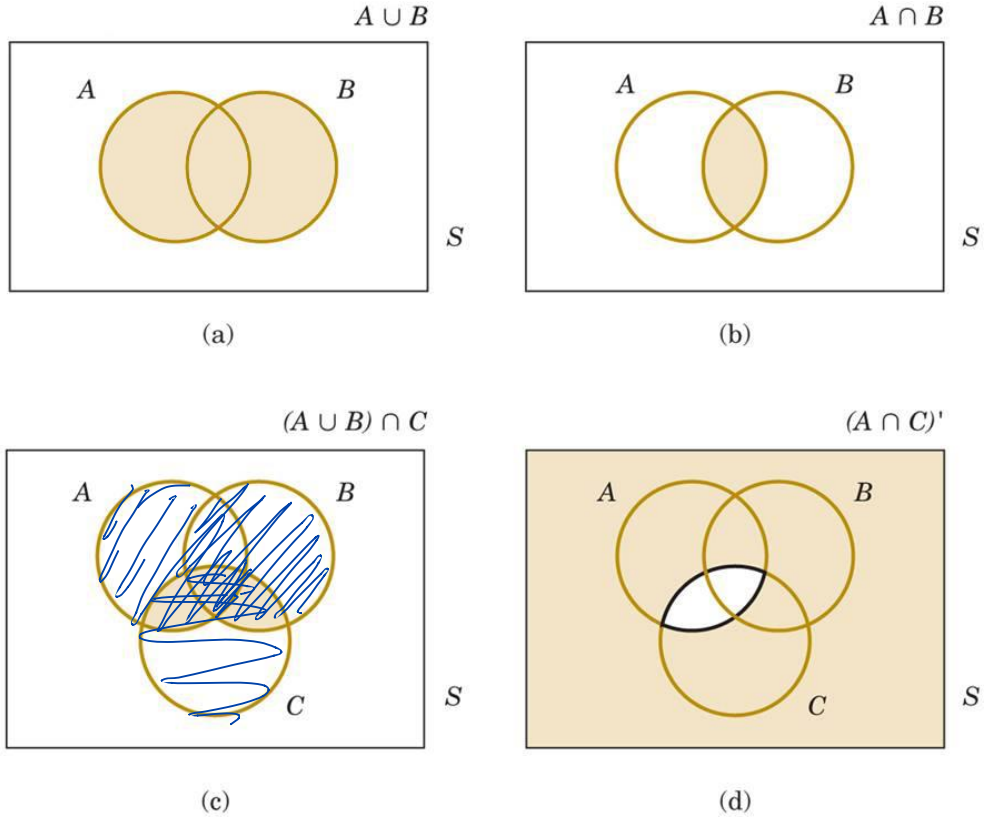
Let $E_1 = \{x \mid 10 \leq x < 12\}$,

Let $E_2 = \{x \mid 11 < x < 15\}$

- Then $E_1 \cup E_2$
- Then $E_1 \cap E_2$
- Then E_1'
- Then $E_1' \cap E_2$

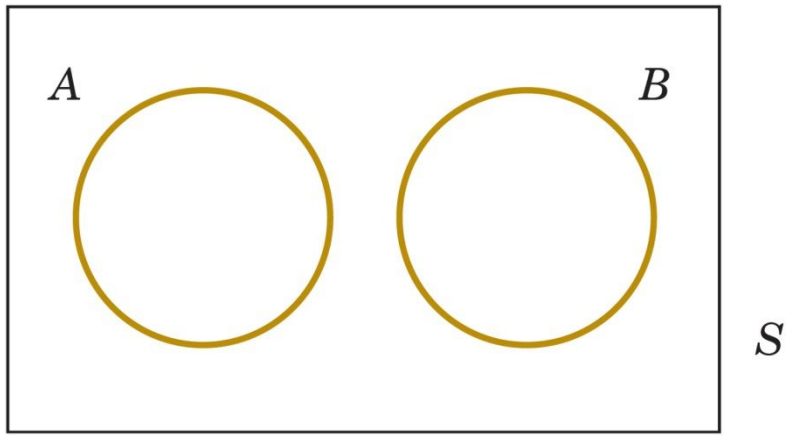
Venn Diagrams

- Events *A* & *B* contain their respective outcomes. The shaded regions indicate the event relation of each diagram.



Mutually Exclusive Events

- Events *A* and *B* are mutually exclusive because they share no common outcomes.
- The occurrence of one event precludes (excludes) the occurrence of the other.
- Symbolically, $A \cap B = \varnothing$



Events - Laws

- Commutative law (event order is unimportant):
  - $A \cap B = B \cap A$ and $A \cup B = B \cup A$
- Distributive law (like in algebra):
  - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- Associative law (like in algebra):
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$

Events - Laws

- DeMorgan's law:
  - $(A \cup B)' = A' \cap B'$ The complement of the union is the intersection of the complements.
  - $(A \cap B)' = A' \cup B'$ The complement of the intersection is the union of the complements.

- Complement law:

$$(A')' = A$$

2.3 Which of the following events are equal?

(a) $A = \{1, 3\}$;

(b) $B = \{x \mid x \text{ is a number on a die}\}$;

(c) $C = \{x \mid x^2 - 4x + 3 = 0\}$;

(d) $D = \{x \mid x \text{ is the number of heads when six coins are tossed}\}$.

2.14 If $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{2, 3, 4, 5\}$, and $D = \{1, 6, 7\}$, list the elements of the sets corresponding to the following events:

(a) $A \cup C$;

(b) $A \cap B$;

(c) C';

(d) $(C' \cap D) \cup B$;

(e) $(S \cap C)'$;

(f) $A \cap C \cap D'$.

2.3 Counting Sample Points

- There are several counting techniques, used to determine the number of outcomes in events:

1. Multiplication rule
2. Permutation rule
3. Partitioning
4. Combination rule

- Each has its special purpose that must be applied properly – the right tool for the right job.

Counting – Multiplication Rule

- Multiplication rule:
  - Let an operation consist of *k* steps and there are
    - n_1 ways of completing step 1,
    - n_2 ways of completing step 2, … and
    - n_k ways of completing step *k*.
  - Then, the total number of ways to perform *k* steps is:
    - $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

Multiplication- Examples

- Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?
- An ice-cream store offers three types of cones and 31 flavors. How many different single-scoop ice-cream cones is it possible to buy at this store?

Counting- Permutations

- Permutation Rule: If we are interested in a sample space that contains as elements all possible orders or arrangements of a group of objects.
- Permutation: is an arrangement of all parts of a set of objects

1. The number of permutations of n distinct objects is (n factorial):

$$n! = n(n-1)(n-2) \ldots \ldots (3)(2)(1)$$

If S= {*a, b, c, d*}, then the number of permutations of the four letters will be

$$4! = 24$$

Counting- Permutations

2. The number of permutations of n distinct objects taken r at a time is:

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Example 2.18: In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

***Solution*:** Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

$$ {}_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800. $$

Counting- Permutations- Example

- A witness to hit and run accident told the police that the license number contained the letters RLS followed by 4 digits, the first of which is 4. If the witness cannot recall the last 3 digits, but he is certain that all 4 digits are different

- RLS4 _ _ _
- 0,1,2,3,4,5,6,7,8,9
- Because of the 3 digits are different 4 will be out
- Therefore the answere will be 9!/(9-3)!=(9x8x7x6!)/(6!) = 504

Counting- Permutations- Similar Items Partitioning

3. The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind,……, n_r of a rth kind is:

$$\frac{n!}{(n_1!n_2!....n_k!)}$$ Where $n_1 + n_2 +……+n_r = n$

Any intersection of every possible pair of cells (r) is the empty set Ø

Union is the S

Example 2.20: In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

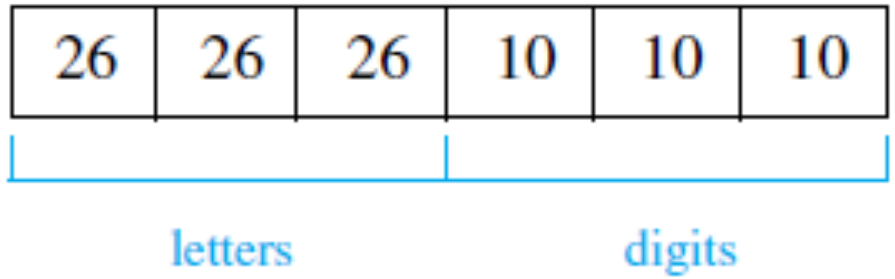
$$\frac{10!}{1!\ 2!\ 4!\ 3!} = 12,600.$$

Counting- Permutations Examples

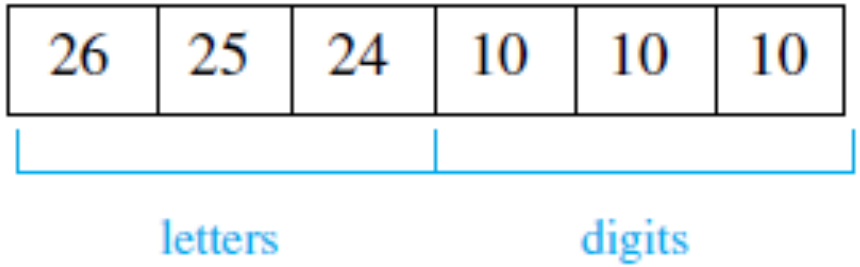
- In a certain state, automobile license plates display three letters followed by three digits. How many such plates are possible if repetition of the letters:

(a) is allowed?

(b) is not allowed?



$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$$



$$26 \times 25 \times 24 \times 10 \times 10 \times 10 = 15{,}600{,}000$$

Concept Question: DNA

DNA is made of sequences of nucleotides: A, C, G, T.

How many DNA sequences of length 3 are there?

(i) 12 (ii) 24 (iii) 64 (iv) 81

answer: (iii) $4 \times 4 \times 4 = 64$

How many DNA sequences of length 3 are there with no repeats?

(i) 12 (ii) 24 (iii) 64 (iv) 81

answer: (ii) $4 \times 3 \times 2 = 24$

I won't wear green and red together; I think black or denim goes with anything; Here is my wardrobe.

Shirts: 3B, 3R, 2G; sweaters 1B, 2R, 1G; pants 2D,2B.

How many different outfits can I wear?

Counting- Permutations Examples

A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if

(a) there are no restrictions;

(b) A will serve only if he is president;

(c) B and C will serve together or not at all;

(d) D and E will not serve together?

Solution: (a) The total number of choices of officers, without any restrictions, is

$$_{50}P_2 = \frac{50!}{48!} = (50)(49) = 2450.$$

(b) Since A will serve only if he is president, we have two situations here: (i) A is selected as the president, which yields 49 possible outcomes for the treasurer's position, or (ii) officers are selected from the remaining 49 people without A, which has the number of choices $_{49}P_2 = (49)(48) = 2352$. Therefore, the total number of choices is $49 + 2352 = 2401$.

(c) The number of selections when B and C serve together is 2. The number of selections when both B and C are not chosen is $_{48}P_2 = 2256$. Therefore, the total number of choices in this situation is $2 + 2256 = 2258$.

(d) The number of selections when D serves as an officer but not E is $(2)(48) = 96$, where 2 is the number of positions D can take and 48 is the number of selections of the other officer from the remaining people in the club except E. The number of selections when E serves as an officer but not D is also $(2)(48) = 96$. The number of selections when both D and E are not chosen is $_{48}P_2 = 2256$. Therefore, the total number of choices is $(2)(96) + 2256 = 2448$. This problem also has another short solution: Since D and E can only serve together in 2 ways, the answer is $2450 - 2 = 2448$.

Counting – Combinations

- A combination is a selection of *r* items from a set of *n* where **order does not matter**.
- If *S* = {*a, b, c*}, *n* =3, then
  - If *r* = 3, there is 1 combination, namely: *abc*
  - If *r* = 2, there are 3 combinations, namely *ab*, *ac*, and *bc*
- # of permutations ≥ # of combinations
- Since order does not matter with combinations, we are dividing the # of permutations by *r*!, where *r*! is the # of arrangements of *r* elements.

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Counting – Combinations

abc	acb	bac	bca	cab	cba	$\{a, b, c\}$
abd	adb	bad	bda	dab	dba	$\{a, b, d\}$
acd	adc	cad	cda	dac	dca	$\{a, c, d\}$
bcd	bdc	cbd	cdb	dbc	dcb	$\{b, c, d\}$

Permutations: ${}_4P_3$ Combinations: ${}_4C_3$

- Example: A club has nine members. In how many ways can a committee of three be chosen from the members of this club?
- Example: A bin of 50 parts contains 3 defectives and 47 non-defective parts. A sample of 6 parts is selected from the 50 without replacement. **How many samples of size 6 contain 2 defective parts?**
- First, how many ways are there for selecting 2 parts from the 3 defective parts?

$$C_2^3 = \frac{3!}{2! \cdot 1!} = 3 \text{ different ways}$$

Counting – Combinations

- Now, how many ways are there for selecting 4 parts from the 47 non-defective parts?

$$C_4^{47} = \frac{47!}{4! \cdot 43!} = \frac{47 \cdot 46 \cdot 45 \cdot 44 \cdot 43!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 43!} = 178{,}365 \text{ different ways}$$

  - So the number of ways to obtain 2 from 3 defectives, and 4 from 47 non-defectives:

$$C_2^3 C_4^{47} = 3 \cdot 178{,}365 = 535{,}095 \text{ different ways}$$

Counting – Combinations

- A committee of seven—consisting of a chairman, a vice chairman, a secretary, and four other members—is to be chosen from a class of 20 students. In how many ways can this committee be chosen?

2.33 If a multiple-choice test consists of 5 questions, each with 4 possible answers of which only 1 is correct,

(a) in how many different ways can a student check off one answer to each question?

(b) in how many ways can a student check off one answer to each question and get all the answers wrong?

2.39 In a regional spelling bee, the 8 finalists consist of 3 boys and 5 girls. Find the number of sample points in the sample space S for the number of possible orders at the conclusion of the contest for

(a) all 8 finalists;

(b) the first 3 positions.

2.4 Probability of an Event

- "I have fifty-fifty chance of getting an even number when a die is tossed"
- Probability = Relative Frequency
- It is the chance of something happening out of a defined number of opportunities.
- $\text{Relative Frequency} = \frac{\text{Frequency}}{\text{Total}}$
- For each element, a weight or "probability" between 0 and 1, will be assigned
- The total weights (i.e., probabilities) = 1

2.4 Probability of an Event

- To find the probability of an event *A*, we sum all the probabilities assigned to the sample points in *A*. This sum is called the **probability** of *A* and is denoted by *P*(*A*).
- $0 \le P(A) \le 1, \quad P(\emptyset) = 0, \quad P(S) = 1$
- If $A_1, A_2, A_3, \ldots\ldots$is a sequence of mutually exclusive events, then:

  $P(A_1 \cup A_2 \cup A_3 \cup \ldots..) = P(A_1) + P(A_2) + P(A_3) + \ldots\ldots.$
- If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event *A*, then the probability of event A is:

  $P(A) = n/N$

2.4 Probability of an Event

Example 2.24: A coin is tossed twice. What is the probability that at least 1 head occurs?

***Solution*:** The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}$$

If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of ω to each sample point. Then $4\omega = 1$, or $\omega = 1/4$. If A represents the event of at least 1 head occurring, then

$$A = \{HH, HT, TH\} \text{ and } P(A) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

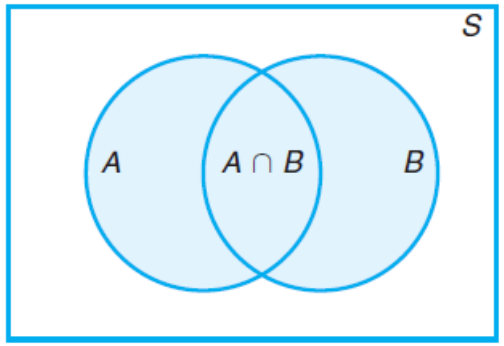
- Example: From a bin of 50 parts, 6 parts are selected randomly without replacement. The bin contains 3 defective parts and 47 non-defective parts. What is the probability that exactly 2 defective parts are selected in the sample?

$$C_2^3 C_4^{47} = 3 \cdot 178{,}365 = 535{,}095 \text{ different ways}$$

$$\binom{50}{6} = \frac{50!}{6!\,44!} = 15{,}890{,}700$$

$$\frac{535{,}095}{15{,}890{,}700} = 0.034$$

2.5 Additive Rules



- Rule 1: If A and B are any two events, then

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

  Proof? Venn Diagram

- If A and B are mutually exclusive, then

  $$P(A \cup B) = P(A) + P(B)$$

- A collection of events $\{A1, A2, \ldots, An\}$ of a sample space S is called a **partition** of S if $A1, A2, \ldots, An$ are mutually exclusive and $A1 \cup A2 \cup \cdots \cup An = S$. Thus, we have

  $$P(A1 \cup A2 \cup \cdots \cup An) = P(A1) + P(A2) + \cdots + P(An) = P(S) = 1.$$

2.5 Additive Rules

- Rule 2: For three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

- Rule 3: If A and A' are complementary events, then

$$P(A) + P(A') = 1$$

- **Example 2.31:** If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colors?

Let G, W, R, and B be the events that a buyer selects, respectively, a green, white, red, or blue automobile. Since these four events are mutually exclusive, the probability is

$$\begin{aligned} P(G \cup W \cup R \cup B) &= P(G) + P(W) + P(R) + P(B) \\ &= 0.09 + 0.15 + 0.21 + 0.23 = 0.68. \end{aligned}$$

2.5 Additive Rules

- **Example 2.33:** Suppose the manufacturer's specifications for the length of a certain type of computer cable are 2000 ± 10 millimeters. In this industry, it is known that small cable is just as likely to be defective (not meeting specifications) as large cable. That is, the probability of randomly producing a cable with length exceeding 2010 millimeters is equal to the probability of producing a cable with length smaller than 1990 millimeters. The probability that the production procedure meets specifications is known to be 0.99.

a) What is the probability that a cable selected randomly is too large?

b) What is the probability that a randomly selected cable is larger than 1990 millimeters?

***Solution*:** Let M be the event that a cable meets specifications. Let S and L be the events that the cable is too small and too large, respectively. Then

(a) $P(M) = 0.99$ and $P(S) = P(L) = (1 - 0.99)/2 = 0.005$.

(b) Denoting by X the length of a randomly selected cable, we have

$$P(1990 \leq X \leq 2010) = P(M) = 0.99.$$

Since $P(X \geq 2010) = P(L) = 0.005$,

$$P(X \geq 1990) = P(M) + P(L) = 0.995.$$

This also can be solved by using Theorem 2.9:

$$P(X \geq 1990) + P(X < 1990) = 1.$$

Thus, $P(X \geq 1990) = 1 - P(S) = 1 - 0.005 = 0.995$.

2.66 Factory workers are constantly encouraged to practice zero tolerance when it comes to accidents in factories. Accidents can occur because the working environment or conditions themselves are unsafe. On the other hand, accidents can occur due to carelessness or so-called human error. In addition, the worker's shift, 7:00 A.M.–3:00 P.M. (day shift), 3:00 P.M.–11:00 P.M. (evening shift), or 11:00 P.M.–7:00 A.M. (graveyard shift), may be a factor. During the last year, 300 accidents have occurred. The percentages of the accidents for the condition combinations are as follows:

Shift	Unsafe Conditions	Human Error
Day	5%	32%
Evening	6%	25%
Graveyard	2%	30%

If an accident report is selected randomly from the 300 reports,

(a) what is the probability that the accident occurred on the graveyard shift?

(b) what is the probability that the accident occurred due to human error?

(c) what is the probability that the accident occurred due to unsafe conditions?

(d) what is the probability that the accident occurred on either the evening or the graveyard shift?

(a) $0.02 + 0.30 = 0.32 = 32\%$;

(b) $0.32 + 0.25 + 0.30 = 0.87 = 87\%$;

(c) $0.05 + 0.06 + 0.02 = 0.13 = 13\%$;

(d) $1 - 0.05 - 0.32 = 0.63 = 63\%$

2.6 Conditional Probability, Independence, and the Product Rule

2.6 Conditional Probability

- $P(B \mid A)$ is the probability of event B occurring, given that event A has already occurred.
- The conditional probability of an event B given an event A, denoted as $P(B \mid A)$, is:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \text{ for } P(A) > 0.$$

- From a relative frequency perspective of n equally likely outcomes:
  - $P(A)$ = (number of outcomes in A) / n
  - $P(A \cap B)$ = (number of outcomes in $A \cap B$) / n
  - $P(B \mid A)$ = number of outcomes in $A \cap B$ / number of outcomes in A

2.6 Conditional Probability

- **Example 2.34:** The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane:
- (a) arrives on time, given that it departed on time, and
- (b) departed on time, given that it has arrived on time.

$$P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

$$P(A|D') = \frac{P(A \cap D')}{P(D')} = \frac{0.82 - 0.78}{0.17} = 0.24.$$

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95.$$

A number from the sample space S = {2, 3, 4, 5, 6, 7, 8, 9} is randomly selected. Given the defined events A and B,

A: selected number is odd, and

B: selected number is a multiple of 3

find the following probabilities.

a) P(B) b) P(A and B) c) P(B/A)

a) B = {3, 6, 9} P(B) = 3/8

b) P(A and B) = $P(\{3, 5, 7, 9\} \cap \{3, 6, 9\})$

$= P(\{3, 9\}) = 2/8 = 1/4$

c) Probability of B given A has occurred:

$$P(B/A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{1/4}{4/8} = 1/2$$

2.6 Conditional Probability

- Example: Given a family with two children, find the probability that both are boys, given that at least one is a boy.
- S= {gg,gb,bg,bb}
- A: at least one boy A={gb,bg,bb}
- B: both are boys B= {bb}
- P(A ∩B)= P({gb,bg,bb) ∩ {bb})= P({bb})= ¼
- P(A)= P({gb,bg,bb}) = ¾
- $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{3/4}$= 1/3

2.6 Conditional Probability

- There are 4 probabilities conditioned on flaws in the below table.
- Find:
- P(F)=
- P(D \ F)=
- P($\bar{D}$\ F)=
- $P(D \setminus \bar{F}) =$
- $P(\bar{D} \setminus \bar{F}) =$

Parts Classified			
Defective	Surface Flaws		Total
	Yes (F)	No (F')	
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400

$$P(F) = 40/400 \text{ and } P(D) = 28/400$$

$$P(D \mid F) = P(D \cap F)/P(F) = \frac{10}{400} / \frac{40}{400} = \frac{10}{40}$$

$$P(D' \mid F) = P(D' \cap F)/P(F) = \frac{30}{400} / \frac{40}{400} = \frac{30}{40}$$

$$P(D \mid F') = P(D \cap F')/P(F') = \frac{18}{400} / \frac{360}{400} = \frac{18}{360}$$

$$P(D' \mid F') = P(D' \cap F')/P(F') = \frac{342}{400} / \frac{360}{400} = \frac{342}{360}$$

- **Random means each item is equally likely to be chosen. If more than one item is sampled, random means that every sampling outcome is equally likely.**

Multiplication Rule

- The conditional probability can be rewritten to generalize a multiplication rule.

$$P(A \cap B) = P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$$

- The last expression is obtained by exchanging the roles of *A* and *B*.

Example 2.36: Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

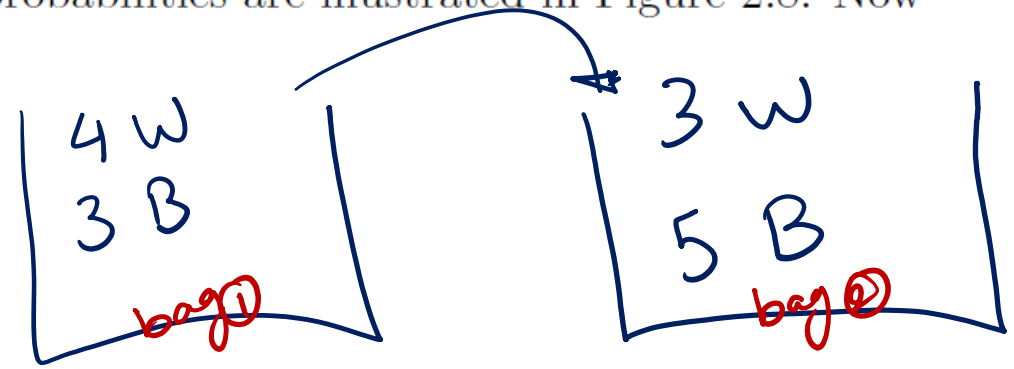
$$P(A \cap B) = \left(\frac{1}{4}\right)\left(\frac{4}{19}\right) = \frac{1}{19}$$

Example 2.37: One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Solution: Let B_1, B_2, and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events $B_1 \cap B_2$ and $W_1 \cap B_2$. The various possibilities and their probabilities are illustrated in Figure 2.8. Now

Example 2.37: One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Solution: Let B_1, B_2, and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events $B_1 \cap B_2$ and $W_1 \cap B_2$. The various possibilities and their probabilities are illustrated in Figure 2.8. Now



- Solution: Let B1, B2, and W1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events B1 ∩ B2 and W1 ∩ B2. The various possibilities and their probabilities are illustrated in Figure 2.8. Now P[(B1 ∩ B2) or (W1 ∩ B2)] = P(B1 ∩ B2) + P(W1 ∩ B2) =
- P(B1)P(B2|B1) + P(W1)P(B2|W1) =
- (3/ 7) (6/ 9) + (4/ 7) (5/ 9) = 38/ 63.

Event Independence

- Two events are independent if any one of the following equivalent statements is true:
  1. $P(A \mid B) = P(A)$
  2. $P(B \mid A) = P(B)$
  3. $P(A \cap B) = P(A) \cdot P(B)$
- This means that occurrence of one event has no impact on the probability of occurrence of the other event.

2.6 Conditional Probability, Independence, and the Product Rule

Example: The probability that a part made in the 1st stage of a machining operation meets specifications is 0.90. The probability that it meets specifications in the 2nd stage, given that it met specifications in the first stage is 0.95.

What is the probability that both stages meet specifications?

- Let A and B denote the events that the part has met 1st and 2nd stage specifications, respectively.
- $P(A \cap B) = P(B \mid A) \cdot P(A) = 0.95 \cdot 0.90 = 0.855$

Example: Flaws and Functions

- Table 1 provides an example of 400 parts classified by surface flaws and as (functionally) defective. Suppose that the situation is different and follows Table 2. Let *F* denote the event that the part has surface flaws. Let *D* denote the event that the part is defective. Find P(D).
- The data shows whether the events are independent or not.

TABLE 1 Parts Classified

	Surface Flaws		
Defective	Yes (F)	No (F')	Total
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400
	$P(D\|F) =$	10/40 =	0.25
	$P(D) =$	28/400 =	0.10
			not same
	Events D & F are **dependent**		

TABLE 2 Parts Classified (data chg'd)

	Surface Flaws		
Defective	Yes (F)	No (F')	Total
Yes (D)	2	18	20
No (D')	38	342	380
Total	40	360	400
	$P(D\|F) =$	2/40 =	0.05
	$P(D) =$	20/400 =	0.05
			same
	Events D & F are **independent**		

Independence with Multiple Events

The events $E_1, E_2, \ldots, E_k$ are independent if and only if, for any subset of these events:

$$P(E_{i1} \cap E_{i2} \cap \ldots, \cap E_{ik}) = P(E_{i1}) \cdot P(E_{i2}) \cdot \ldots \cdot P(E_{ik})$$

- Example: Assume the probability that a wafer contains a large particle of contamination is 0.01 and that the wafers are independent; that is, the probability that a wafer contains a large particle does not depend on the characteristics of any of the other wafers. If 15 wafers are analyzed, what is the probability that no large particles are found?

  Solution:

  Let E_i denote the event that the i^{th} wafer contains no large particles,

  i = 1, 2, …,15.

  Then , $P(E_i) = 0.99$.

  The required probability is $P(E_1 \cap E_2 \cap \ldots \cap E_{15})$.

  From the assumption of independence,

  $$P(E_1 \cap E_2 \cap \ldots \cap E_{15}) = P(E_1)\cdot P(E_2)\cdot\ldots\cdot P(E_{15})$$
  $$= (0.99)^{15}$$
  $$= 0.86.$$

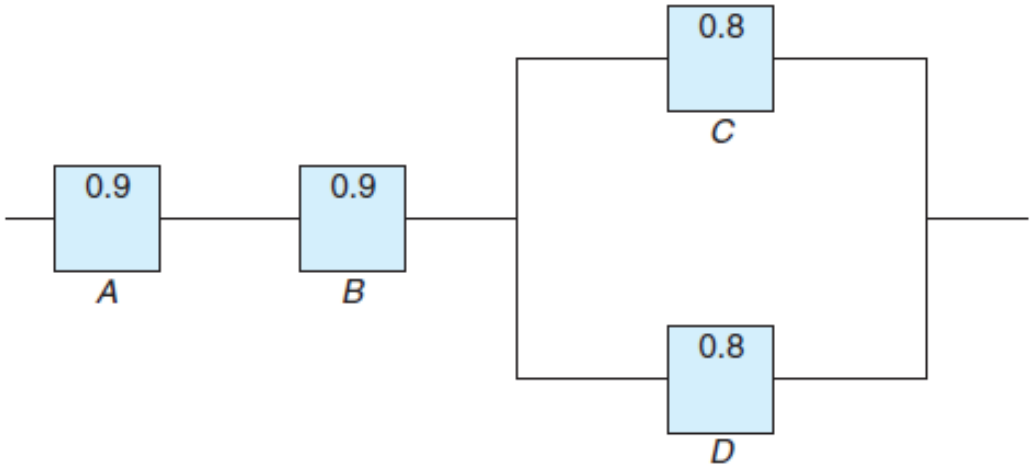
Example 2.38:

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

$$P(A \cap B) = P(A)P(B) = (0.98)(0.92) = 0.9016.$$

Example 2.39:

An electrical system consists of four components as illustrated in Figure 2.9. The system works if components A and B work and either of the components C or D works. The reliability (probability of working) of each component is also shown in Figure 2.9. Find the probability that (a) the entire system works and (b) the component C does not work, given that the entire system works. Assume that the four components work independently.



$$P[A \cap B \cap (C \cup D)] = P(A)P(B)P(C \cup D) = P(A)P(B)[1 - P(C' \cap D')]$$
$$= P(A)P(B)[1 - P(C')P(D')]$$
$$= (0.9)(0.9)[1 - (1 - 0.8)(1 - 0.8)] = 0.7776.$$

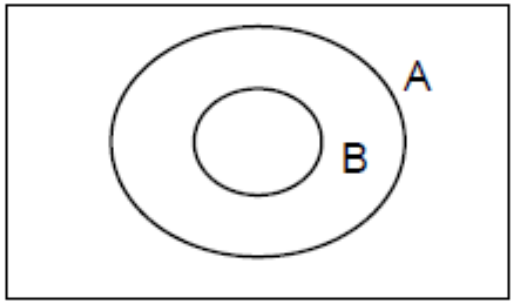
The equalities above hold because of the independence among the four components.

(b) To calculate the conditional probability in this case, notice that

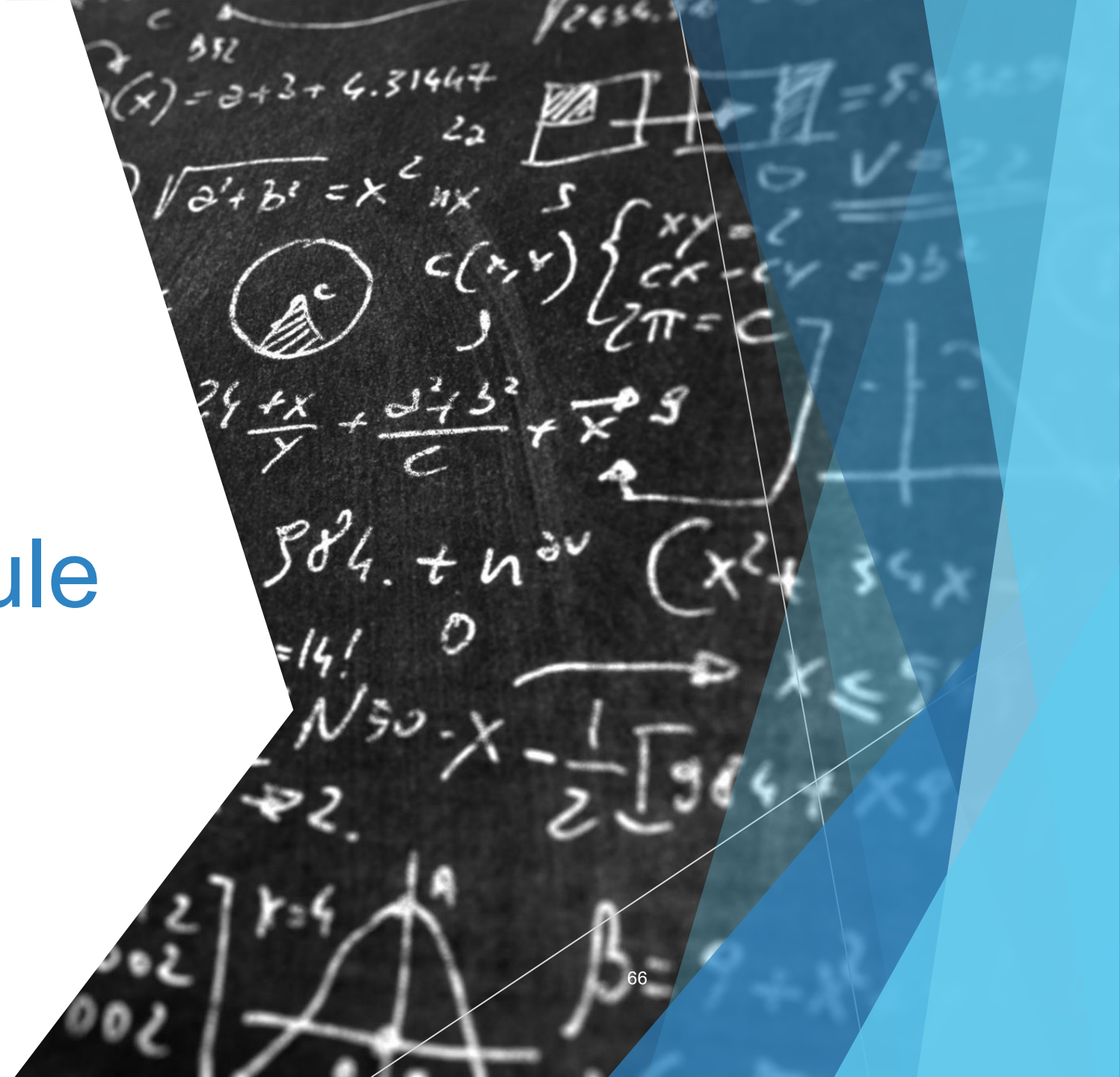
$$P = \frac{P(\text{the system works but } C \text{ does not work})}{P(\text{the system works})}$$
$$= \frac{P(A \cap B \cap C' \cap D)}{P(\text{the system works})} = \frac{(0.9)(0.9)(1 - 0.8)(0.8)}{0.7776} = 0.1667.$$

Exercises

- If $P(A \mid B) = 1$, must $A = B$? Draw a Venn diagram to explain your answer.
- No, if $B \subset A$, then $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$



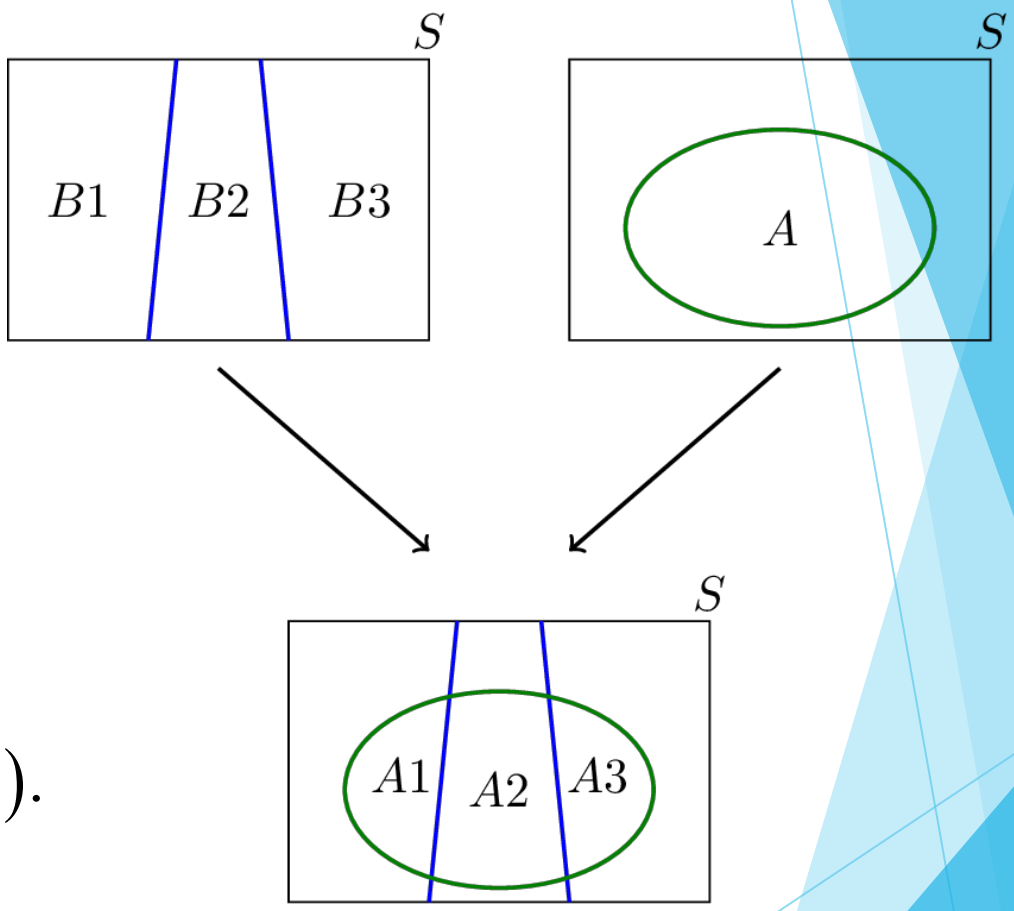
2.7 Baye's Rule



Law of total Probability

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of sample space *S* such that the $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event *A* of *S*,

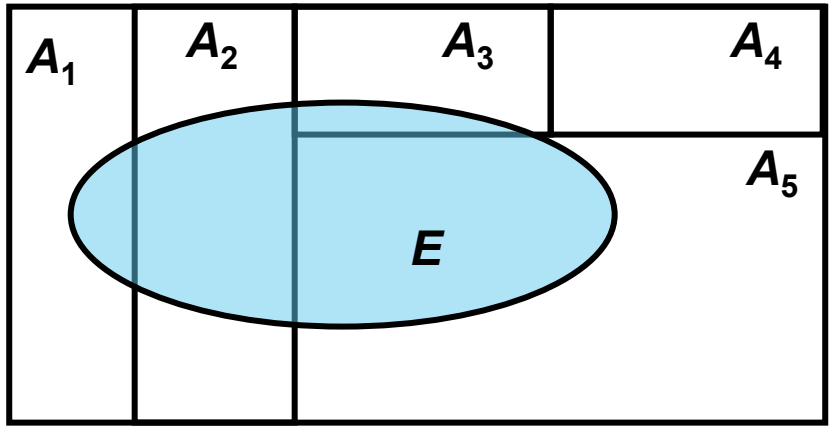
$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i) P(A \mid B_i).$$



Theorem of Total Probability

If $A_1, A_2, A_3, \ldots A_n$, represents a partition of a sample space S, and $E \subset S$ represents an arbitrary event, the theorem states that

$$P(E) = P(A_1)P(E \mid A_1) + P(A_2)P(E \mid A_2) + \ldots + P(A_n)P(E \mid A_n)$$



Theorem of Total Probability

- If structural damage (*D*) to a building can only be caused by three events: *fire* (*F*), strong *wind* (*W*), or *earthquake* (*E*), then *D* will depend on whether *F*, *W*, or *E* has occurred, and the likelihood of occurrence of *F*, *W*, and *E.*
- If we assume further that *F*, *W*, and *E* are collectively exhaustive and mutually exclusive events (partitions), then the probability of damage to the building can be computed as

$$\mathrm{P}(D) = \mathrm{P}(D \mid F)\mathrm{P}(F) + \mathrm{P}(D \mid W)\mathrm{P}(W) + \mathrm{P}(D \mid E)\mathrm{P}(E)$$

- Each term in the right-hand side of the above equation calculates the probability of damage given that *fire*, *wind*, or *earthquake* has occurred. The concept of above equation is called the theorem of total probability

Bayes Theorem

(Bayes' Rule) If the events $B_1, B_2, ..., B_k$ constitute a partition of the sample space *S* such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event *A* in *S* such that $P(A) \neq 0$,

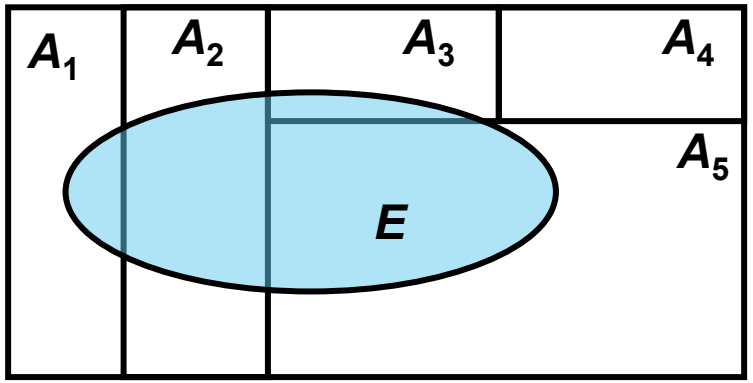
$$P(B_r \mid A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A \mid B_r)}{\sum_{i=1}^{k} P(B_i)P(A \mid B_i)} \quad \text{for} \quad r = 1, 2, ..., k.$$

Theorem of Total Probability

- **Bayes' Theorem:** Is useful in computing the reverse probability of the type P(A_i | E), for I = 1, 2, …,n. The reverse probability can be computed as

$$P(A_i \mid E) = \frac{P(A_i)P(E \mid A_i)}{P(A_1)P(E \mid A_1) + P(A_2)P(E \mid A_2) + \ldots + P(A_n)P(E \mid A_n)}$$

$$= \frac{P(A_i)P(E \mid A_i)}{P(E)}$$



Theorem of Total Probability

- If structural damage (*D*) to a building can only be caused by three events: *fire* (*F*), strong *wind* (*W*), or *earthquake* (*E*), then *D* will depend on whether *F*, *W*, or *E* has occurred, and the likelihood of occurrence of *F*, *W*, and *E.*
- If we assume that D (damage) has occurred what is the probability that it was from *F*, *W*, or *E*

$$P(F \mid D) = \frac{P(F)P(D \mid F)}{P(D)} =$$

$$P(W \mid D) = \frac{P(W)P(D \mid W)}{P(D)}$$

$$P(E \mid D) = \frac{P(E)P(D \mid E)}{P(D)} =$$

Example 1: Theorem of Total Probability

Example: Structural Damage of a Building

A building can have structural damage (D) during its lifetime only from fire (F), strong winds (W), and strong earthquake (E). The probabilities of structural damage to the building if these events occur are estimated to 0.005, 0.01, and 0.05, respectively. The probabilities of occurrence of F, W, and E during the life of the building are 0.5, 0.3, and 0.2, respectively.

a. What is the probability that the building will suffer structural damage during its life?
b. If the building has suffered structural damage, what is the probability that it was caused by F? By W? By E?

Summary: $P(D \mid F) = 0.005$, $P(D \mid W) = 0.01$, $P(D \mid E) = 0.05$

$P(F) = 0.5$, $P(W) = 0.3$, $P(E) = 0.2$

Example 1: Theorem of Total Probability

a. Using the theorem of total probability, the probability that the building will suffer structural damage during its life is:

$$\begin{aligned} P(D) &= P(F)P(D \mid F) + P(W)P(D \mid W) + P(E)P(D \mid E) \\ &= 0.5 \times 0.005 + 0.3 \times 0.01 + 0.2 \times 0.05 = 0.0155 \end{aligned}$$

b. Using Bayes' theorem, the probabilities that the structural damage was caused by *F*, *W*, or *E* are:

$$P(F \mid D) = \frac{P(F)P(D \mid F)}{P(D)} = \frac{0.5 \times 0.005}{0.0155} = 0.16$$

$$P(W \mid D) = \frac{P(W)P(D \mid W)}{P(D)} = \frac{0.3 \times 0.01}{0.0155} = 0.19$$

$$P(E \mid D) = \frac{P(E)P(D \mid E)}{P(D)} = \frac{0.2 \times 0.05}{0.0155} = 0.65$$

Example 2: Theorem of Total Probability

Defective Products in Production Lines:

Consider a factory with three production lines, L_1, L_2, and L_3. Products are either defective (D) or non-defective (ND). The following probabilities are given:

$$P(D \mid L_1) = 0.1 \qquad P(D \mid L_2) = 0.1 \qquad P(D \mid L_3) = 0.2$$

Assuming 20%, 30%, and 50% of the components are manufactured by lines 1, 2, and 3, the probability of defective components is

$$P(L_1) = (0.2) \qquad P(L_2) = (0.3) \qquad P(L_3) = (0.5)$$

$$\begin{aligned} P(D) &= P(L_1)P(D \mid L_1) + P(L_2)P(D \mid L_2) + P(L_3)P(D \mid L_3) \\ &= 0.1 \times 0.2 + 0.1 \times 0.3 + 0.2 \times 0.5 = 0.15 \end{aligned}$$

Example: Bayes' Theorem and Probability Trees

Structural Damage of a Building: A building can have structural damage (*D*) during its lifetime only from fire (*F*), strong winds (*W*), and strong earthquake (*E*). The probabilities of structural damage to the building if these events occur are estimated to 0.005. 0.01, and 0.05, respectively. The probabilities of occurrence of *F*, *W*, and *E* during the life of the building are 0.5, 0.3, and 0.2, respectively.

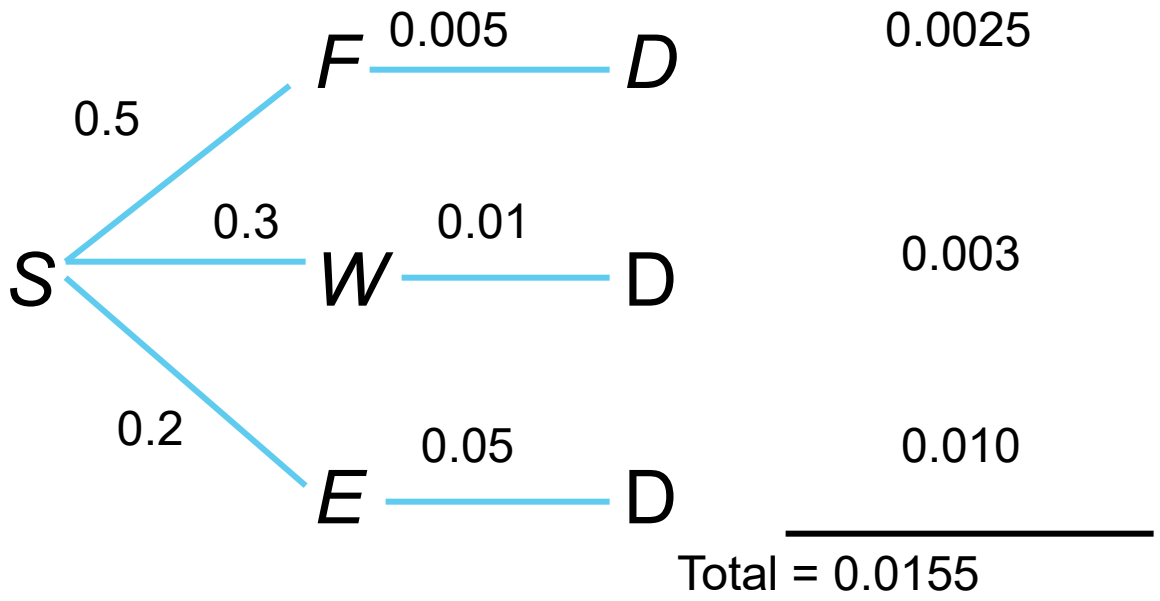
a. What is the probability that the building will suffer structural damage during its life?

b. If the building has suffered structural damage, what is the probability that it was caused by *F*? By *W*? By E?

(previously calculated these without trees SLIDE 71, now use the probability tree to solve the example)

Example: Bayes' Theorem and Probability Trees

$$\mathrm{P}(D \mid F) = 0.005$$
$$\mathrm{P}(D \mid W) = 0.01$$
$$\mathrm{P}(D \mid E) = 0.05$$
$$\mathrm{P}(F) = 0.5$$
$$\mathrm{P}(W) = 0.3$$
$$\mathrm{P}(E) = 0.2$$

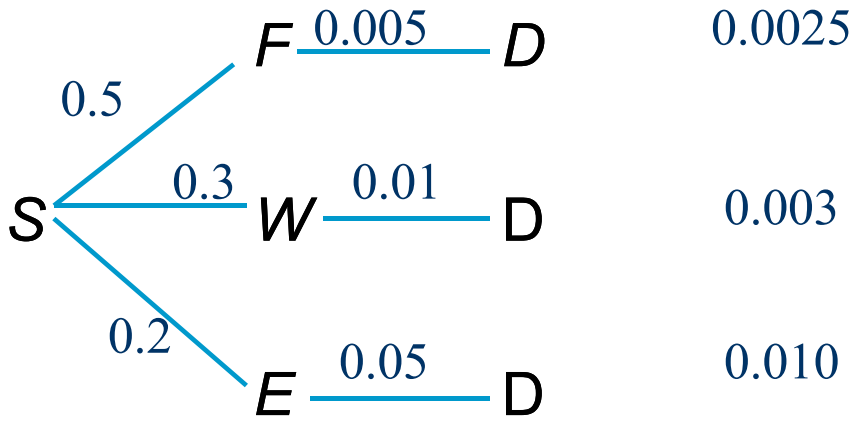


Example: Bayes' Theorem and Probability Trees

a. Using probability tree,

$$P(D) = 0.0025 + 0.003 + 0.010 = 0.0155$$

b. Using probability tree, the probabilities that the structural damage was caused by *F*, by *W*, or by *E* are



0.0025

0.003

0.010

$$P(F \mid D) = \frac{0.0025}{0.0155} = 0.16$$

$$P(W \mid D) = \frac{0.003}{0.0155} = 0.19$$

$$P(E \mid D) = \frac{0.01}{0.0155} = 0.65$$

Example 2.43: A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|P_1) = 0.01, \qquad P(D|P_2) = 0.03, \qquad P(D|P_3) = 0.02,$$

where $P(D|P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

2.103 A truth serum has the property that 90% of the guilty suspects are properly judged while, of course, 10% of the guilty suspects are improperly found innocent. On the other hand, innocent suspects are misjudged 1% of the time. If the suspect was selected from a group of suspects of which only 5% have ever committed a crime, and the serum indicates that he is guilty, what is the probability that he is innocent?